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Comparative Analysis of Some New Runge-Kutta Type Techniques on the Solution of First Order Initial Value Problem in Ordinary Differential Equations

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ABSTRACT

The derivation of numerical methods to deal with differential equations framed from real life problems has been on the rise of which great deal of attention have been drawn towards Runge-kutta methods. In recent times, researchers have explored the derivation of Runge-Kutta methods by introducing higher order derivative (up to the second order) in the terms of Runge-Kutta methods. We have also seen how other types of 'mean' are used as a substitute to the more usually applied arithmetic mean in the derivation Runge Kutta methods. However, in this paper some new Runge-Kutta type methods which border on the use of other types of 'mean' such as harmonic mean, geometric mean or heronian mean with higher derivatives up to the second derivative on a single explicit Runge-Kutta methods which were previously done on different explicit Runge-Kutta methods are constructed, analyzed, implemented and compared. The qualitative features of the methods including the local truncation error, consistency, convergence and stability of the new methods were comparatively analyzed, investigated and established. We demonstrated the validity of the comparisons with four numerical examples. The obtained results were compared with some numerical methods and the exact solutions of the proposed problems.

Keywords: initial value problems; explicit runge-kutta methods; heronian mean; geometric mean; harmonic mean.

INTRODUCTION

A large amount of real-life problems in sciences and engineering can be reduced to mathematical problem that can be solved under certain conditions. These mathematical problems are often times called differential equations. The analytical methods of solution can be used to solve only selected class of differential equations. These differential equations that rule physical systems do not normally operate closed form solutions as a consequence, numerical methods are resorted to solve such differential equations. In this work, we consider one of such differential equations- first order initial value problem differential equation expressed in the form:

$$y'(x) = f(x, y(x)); \quad y(a) = y_0, \quad a \leq x \leq b \quad (1)$$

The development of numerical methods for the solution have turned out to be a very rapid research area in recent decades. Several methods have been developed using the idea of different types of 'mean' such as the geometric mean, heronian mean, centroidal mean, contra-harmonic mean and harmonic mean.

Akanbi M.A. [4] proposed a 3-stage geometric explicit Runge-Kutta methods for singular autonomous initial value problems in ordinary differential equations where geometric mean was incorporated in the classical 3-stage Runge-Kutta methods. A third order harmonic mean for autonomous initial value problem was constructed by Wusu et al. [21] The method was derived based on harmonic mean and was confirmed to be better than any third order of any form of explicit Runge-Kutta methods. This idea was extended to fourth order in [19]. Olaniyan et al. [16] constructed a new Implicit Runge-Kutta method in which heronian mean was used as a basis in the derivation. The paper was found to perform better than the classical 2-Stage Implicit Runge-Kutta methods.

On another note, researchers in recent times discovered that higher derivative terms can be used to enhance the performance of multistage methods. This discovery led to many researchers employing higher derivative terms in the derivations of their methods of which a significant increase in efficiency were achieved.

In an earlier research work of Goeken D. and Johnson O. [11], a 2-stage explicit Runge-Kutta method of order 3 was developed for autonomous Initial Value Problems with the notion of incorporating first derivative in the internal stages of Runge-Kutta method. This method was later extended to fourth and fifth order methods in [12]. Akanbi [3] improved on this research by deriving a two stage multi-derivative explicit Runge-Kutta method involving first and second derivatives which provided better results. Wusu et al. [21] then presented a new class of three stage Runge-Kutta methods with first and second derivatives of which the cost of internal stage evaluations is reduced greatly and there is an appreciable improvement on the attainable order of accuracy of the method. Lately, Olaniyan et al. [18] extended the research works in [21] to a four-stage multiderivative explicit Runge-Kutta method for the solution of first order ordinary differential equations. Several authors such as the ones in ([1], [2], [6], [14], [15]) to mention a few, have developed similar methods based on higher derivatives Runge-Kutta methods.

All the derivations mentioned in the last paragraph are being viewed in arithmetic mean sense, hence the need to explore other types of 'mean' which have been proven over the years to be another and efficient ways of solving O.D.Es. Consequently, in the research work of Olaniyan et al. [17], a combination of higher derivative up to the second derivative and some types of 'mean' such as harmonic mean, geometric mean and heronian mean were incorporated on a single explicit three stage Runge-Kutta method to develop a new Runge-Kutta type techniques. In this paper, some new four stage numerical integration techniques that bother on the use of higher derivatives up to the second derivative and different types of 'mean' in the main formula are constructed to solve linear and nonlinear initial value problems in O.D.Es. Their qualitative features such as the local truncation error, consistency, convergence and stability of the new methods were comparatively analyzed, investigated and established.

MATERIALS AND METHODS

For the numerical integration of, we consider the derivation of schemes having a combination of some types of 'mean'such as geometric mean, harmonic mean or heronian mean with higher derivatives up to the second derivative in a single Runge-Kutta Methods which are of the forms:

$$
\begin{aligned}
y_{n+1} - y_n &= \Phi(y_n; h) \\
k_1 &= hf(y) \\
k_2 &= hf\left(y + ha_{21}k_1 + h^2 a_{22} f f_y + \frac{h^3}{2} a_{23}(f f_y^2 + f^2 f_{yy})\right) \\
k_3 &= hf\left(y + ha_{31}k_1 + ha_{32}k_2 + h^2 a_{33} f f_y + \frac{h^3}{2} a_{34}(f f_y^2 + f^2 f_{yy})\right) \\
k_4 &= hf\left(y + ha_{41}k_1 + ha_{42}k_2 + ha_{43}k_3 + h^2 a_{44} f f_y + \frac{h^3}{2} a_{45}(f f_y^2 + f^2 f_{yy})\right)
\end{aligned}
\tag{2}
$$

Where $\phi(y_n;h)$ is equivalent to any of the following:

$$\Phi_{GMERK}(y_n; h) = c_1\sqrt{k_1k_2} + c_2\sqrt{k_2k_3} + c_3\sqrt{k_3k_4}, \ (3)$$

$$\Phi_{HaMERK}(y_n; h) = c_1\frac{k_1k_2}{k_1+k_2} + c_2\frac{k_2k_3}{k_2+k_3} + c_3\frac{k_3k_4}{k_3+k_4} \qquad (4)$$

and

$$\Phi_{HeMERK}(y_n; h) = \frac{k_1 + 2k_2 + 2k_3 + k_4 + \sqrt{k_1k_2} + \sqrt{k_2k_3} + \sqrt{k_3k_4}}{9} \ (5)$$

DERIVATION OF THE METHODS

To derive these methods, we will obtain the Taylor's series expansions of k_2, k_3, k_4 as follows:

$$\begin{aligned} k_2 = \; & hf + h^2 a_{21} f f_y + \frac{h^3}{2} a_{21}^2 f^2 f_{yy} + h^3 a_{22} f f_y^2 + \frac{h^4}{6} a_{21}^3 f^3 f_{yyy} \\ & + \frac{h^4}{2}(a_{23} + 2a_{21}a_{22}) f^2 f_y f_{yy} + \frac{h^4}{2} a_{23} f f_y^3 + \frac{h^5}{24} a_{21}^4 f^4 f_{yyyy} \\ & + \frac{h^5}{2} a_{21} a_{23} f^3 f_{yy}^2 + \frac{h^5}{2} a_{21}^2 a_{22} f^3 f_y f_{yyy} \\ & + \frac{h^5}{2}(a_{22}^2 + a_{21}a_{23}) f^2 f_y^2 f_{yy}, \end{aligned} \tag{6}$$

$$\begin{aligned} k_3 = \; & hf + h^2 a_{31} f f_y + \frac{h^3}{2} a_{31}^2 f^2 f_{yy} + h^3 (a_{21}a_{32} + a_{33}) f f_y^2 \\ & + \frac{h^4}{6} a_{31}^3 f^3 f_{yyy} + \frac{h^4}{2}(a_{21}^2 a_{32} + a_{34} + 2a_{21}a_{31}a_{32} + 2a_{31}a_{33}) f^2 f_y f_{yy} \\ & + \frac{h^4}{2}(2a_{22}a_{32} + a_{34}) f f_y^3 + \frac{h^5}{24} a_{31}^4 f^4 f_{yyyy} + \frac{h^5}{2}(a_{21}^2 a_{31} a_{32} \\ & + a_{31}a_{34}) f^3 f_{yy}^2 + \frac{h^5}{2}(2a_{22}a_{31}a_{32} + a_{31}a_{34} + a_{21}^2 a_{32}^2 + 2a_{21}a_{32}a_{33} \\ & + a_{33}^2 + 2a_{21}a_{22}a_{32} + a_{23}a_{32}) f^2 f_y^2 f_{yy} + \frac{h^5}{2}(a_{21}a_{32}a_{31}^2 + a_{33}a_{31}^2 \\ & + \frac{1}{3} a_{21}^3 a_{32}) f^3 f_y f_{yyy} + \frac{h^5}{2} a_{23} a_{32} f f_y^4, \end{aligned} \tag{7}$$

and

$$\begin{aligned} k_4 = \; & hf + h^2 a_{41} f f_y + \frac{h^3}{2} a_{41}^2 f^2 f_{yy} + h^3 (a_{21}a_{42} + a_{31}a_{43} + a_{44}) f f_y^2 \\ & + \frac{h^4}{6} a_{41}^3 f^3 f_{yyy} + \frac{h^4}{2}(a_{21}^2 a_{42} + a_{31}^2 a_{43} + a_{45} + 2a_{21}a_{41}a_{42} + 2a_{41}a_{44} \\ & + 2a_{31}a_{41}a_{43}) f^2 f_y f_{yy} + \frac{h^4}{2}(2a_{22}a_{42} + 2a_{21}a_{32}a_{43} + 2a_{33}a_{43} + a_{45}) f f_y^3 \\ & + \frac{h^5}{24} a_{41}^4 f^4 f_{yyyy} + \frac{h^5}{2}(a_{31}^2 a_{41} a_{43} + a_{41}a_{45} + a_{41}a_{42}a_{21}^2) f^3 f_{yy}^2 \\ & + \frac{h^5}{2}(2a_{21}a_{32}a_{41}a_{43} + 2a_{33}a_{41}a_{43} + a_{41}a_{45} + 2a_{22}a_{41}a_{42} + 2a_{21}a_{42}a_{44} \\ & + a_{21}^2 a_{42}^2 + a_{44}^2 + a_{31}^2 a_{43}^2 + 2a_{21}a_{31}a_{42}a_{43} + 2a_{31}a_{43}a_{44} + 2a_{21}a_{32}a_{31}a_{43} \\ & + 2a_{33}a_{31}a_{43} + a_{21}^2 a_{32} a_{43} + a_{34}a_{43} + 2a_{22}a_{42}a_{21} + a_{23}a_{42}) f^2 f_y^2 f_{yy} \\ & + \frac{h^5}{2}\left(\frac{1}{3} a_{42} a_{21}^3 + \frac{1}{3} a_{31}^3 a_{43} + a_{21}a_{42}a_{41}^2 + a_{31}a_{43}a_{41}^2 + a_{44}a_{41}^2\right) f^3 f_y f_{yyy} \\ & + \frac{h^5}{6}(3a_{23}a_{42} + 6a_{22}a_{32}a_{43} + 3a_{34}a_{43}) f f_y^4 \end{aligned} \tag{8}$$

For $\phi_{GMERK}(y_n; h)$, k_1, and the Taylor's series expansions of k_2, k_3 and k_4 are substituted into (3). Then the resulting equation is compared with the Taylor's series expansion of y_{n+1} about (x_n, y_n) up to order $O(h^5)$ to obtain system of 12 equations with 15 unknowns which were solved with the help of some free parameters to obtain corresponding parameters. For computational advantage we will make $\alpha_{22} = \alpha_{33} = \alpha_{43} = 0$. Upon solving these equations, the values of the parameters were obtained and then substituted into the general form to obtain the following method:

$$
\begin{aligned}
y_{n+1} - y_n &= \frac{1}{4}\sqrt{k_1k_2} + \frac{3}{8}\sqrt{k_2k_3} + \frac{3}{8}\sqrt{k_3k_4} \\
k_1 &= hf(y) \\
k_2 &= hf\left(y + \frac{2}{15}hk_1 - \frac{11}{16}h^3\left(ff_y^2 + f^2f_{yy}\right)\right) \\
k_3 &= hf\left(y + \frac{1}{20}hk_1 + \frac{16}{11}h^2ff_y - \frac{6}{5}h^3\left(ff_y^2 + f^2f_{yy}\right)\right) \\
k_4 &= hf\left(y + \frac{1}{15}hk_1 - \frac{9}{11}hk_2 - \frac{3}{16}h^2ff_y + \frac{4}{13}h^3\left(ff_y^2 + f^2f_{yy}\right)\right)
\end{aligned} \quad (9)
$$

In the same manner, to obtain $\phi HaMERK(y_n;h)$, we will substitute k_1 and equations (9-10) in (4) to get a simplified resulting equation which is then compared with the Taylor's series expansion of y_{n+1} about (x_n, y_n) up to order $O(h^5)$ to obtain system of 12 equations with 15 unknowns that were solved with the help of some free parameters.

For convenience and computational advantage, we will set free parameters as follows: $\alpha_{12}=\frac{1}{2}$ and $\alpha_{32} = \alpha_{43} = 0$. Solving the resulting set of equations, corresponding parameters were gotten and substituted in the general form as:

$$
\begin{aligned}
y_{n+1} - y_n &= \frac{k_1k_2}{3(k_1+k_2)} - \frac{2k_2k_3}{(k_2+k_3)} + \frac{5k_3k_4}{3(k_3+k_4)} \\
k_1 &= hf(y) \\
k_2 &= hf\left(y + \frac{1}{2}hk_1 - \frac{1}{2}h^2ff_y + \frac{3}{2}h^3\left(ff_y^2 + f^2f_{yy}\right)\right) \\
k_3 &= hf\left(y + \frac{1}{2}hk_1 - \frac{3}{8}h^2ff_y + \frac{5}{8}h^3\left(ff_y^2 + f^2f_{yy}\right)\right) \\
k_4 &= hf\left(y + \frac{1}{4}hk_1 - \frac{15}{8}hk_2 + \frac{3}{16}h^2ff_y + \frac{7}{16}h^3\left(ff_y^2 + f^2f_{yy}\right)\right)
\end{aligned} \quad \#(10)
$$

Finally, for $\phi HeMERK(y_n;h)$, again we will substitute k_1 and equations (6) in (5). The result obtained is also compared with the Taylor's series expansion of y_{n+1} about (x_n, y_n) up to order $O(h^5)$. Here, we have 12 equations with 12 unknowns which were systematically solved to get the corresponding parameters. The values of the parameters are then substituted into the general form as follows:

$$
\begin{aligned}
y_{n+1} - y_n &= \frac{k_1 + 2k_2 + 2k_3 + k_4 + \sqrt{k_1k_2} + \sqrt{k_2k_3} + \sqrt{k_3k_4}}{9} \\
k_1 &= f(y) \\
k_2 &= f\left(y + \frac{1}{6}hk_1 - \frac{2}{3}h^3\left(ff_y^2 + f^2f_{yy}\right)\right) \\
k_3 &= f\left(y + \frac{5}{12}hk_1 + \frac{5}{3}h^2ff_y - \frac{11}{6}h^3\left(ff_y^2 + f^2f_{yy}\right)\right) \\
k_4 &= f\left(y + \frac{3}{10}hk_1 - \frac{2}{9}hk_2 + \frac{2}{9}h^2ff_y + \frac{5}{18}h^3\left(ff_y^2 + f^2f_{yy}\right)\right)
\end{aligned} \quad \#(11)
$$

Qualitative Features

We will consider some basic features which are very vital to the development of the constructed schemes. These features are local truncation error, consistency, stability and convergence.

Local Truncation Error

Definition 3.1.1 ([13])

The local truncation error T_{n+1} *at* X_{n+1} of the general one step method is given as

$$T_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n), h)$$

Where $y(x_n)$ is the theoretical solution.

The local truncation error of the constructed schemes in compliance with the above definition can be expressed as

$$T_{n+1} = y(x_{n+1}) - y_{n+1}$$

Definition 3.1.2 ([13])

A numerical method is said to be of order $\boldsymbol{p}$ if $\boldsymbol{p}$ is the largest integer for which $T_{n+1} = O(h^{p+1})$ for every n and $p \geq 1$.

Consequently, the local truncation error of the methods constructed are as follows:

4-Stage GMERK

$$T_{n+1} = \frac{h^6}{2160}\left(5ff_y^5 - 12f^2f_y^3f_{yy} + 14f^3f_yf_{yy}^2 - 20f^3f_y^2f_{yyy} - 15f^4f_{yy}f_{yyy} + 9f^4f_yf_{yyyy} - 4f^5f_{yyyyy}\right) \quad (12)$$

4-Stage HaMERK

$$T_{n+1} = \frac{h^6}{7680}\left(7ff_y^5 - 22f^2f_y^3f_{yy} + 31f^3f_yf_{yy}^2 - 28f^3f_y^2f_{yyy} - 33f^4f_{yy}f_{yyy} + 35f^4f_yf_{yyyy} - 9f^5f_{yyyyy}\right) \quad (13)$$

4-Stage HeMERK

$$T_{n+1} = \frac{h^6}{23040}\left(16ff_y^5 - 28f^2f_y^3f_{yy} + 48f^3f_yf_{yy}^2 - 39f^3f_y^2f_{yyy} - 43f^4f_{yy}f_{yyy} + 29f^4f_yf_{yyyy} - 14f^5f_{yyyyy}\right) \quad (14)$$

Theorem 3.1.3 ([13])
Let $f(x,y)$ belongs to $C^4[a,b]$ and let its partial derivatives be bounded and if there exist ***L, M*** some positive constants such that

$$|f(x,y)| < M, \quad \left|\frac{\delta^{i+j}}{\delta x^i \delta y^j}\right| < \frac{L^{i+j}}{M^{i-j}}, \quad i+j < M$$

then in terms of error bound by virtue of Lotkin in Lambert [13], hence the strict upper bound with respect to ***y*** only for the constructed methods are given as follows:

$$|LTE_{GM}| \le \frac{23}{2160}h^5ML^4 + O(h)^6$$

$$|LTE_{HA}| \le \frac{19}{7680}h^5ML^4 + O(h)^6$$

$$|LTE_{HE}| \le \frac{31}{23040}h^5ML^4 + O(h)^6$$

Consistency
Definition 3.2.1 ([13])
A numerical method is said to be consistent with an initial value problem if

$\phi(x,y,0) \equiv f(x,y)$
Thus, a consistent method has at least order one.

Definition 3.1.2 ([13])
A scheme is said to be consistent if the difference equation of the integrating formula exactly approximates the differential equation it intends to solve as the step size approaches zero. In order to establish the consistency property of the proposed method it is sufficient to show that

$$\lim_{h\to 0}\phi(x_n, y_n; h) = f(x_n, y_n)$$

where $\phi(x_n, y_n; h)$ is the increment function of the numerical method.

The consistency of the derived methods was investigated using the above consistency definitions and were all confirmed consistent.

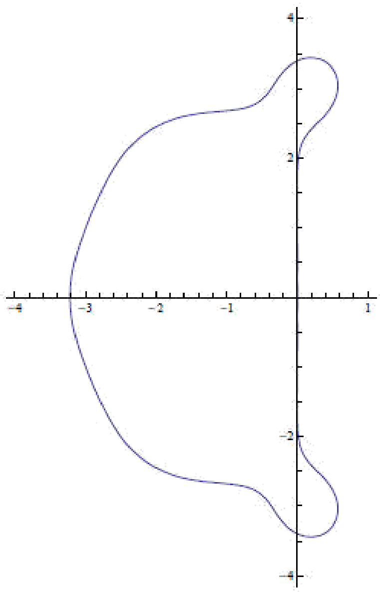

FIGURE 1: Absolute Stability Region of the Proposed Methods.

Stability of the Derived Schemes

The stability of numerical methods for solving an IVP in ODE can be analyzed using the linear test problem $y'=\lambda y$ proposed in [7] , where the solution is $y=e^{\lambda y}$ and λ a complex constant. Applying $\phi_{GMERK}(y_n;h)$, $\phi HaMERK(y_n;h)$, and $\phi HeMERK(y_n;h)$, to the linear test problem and allowing $z=\lambda h$. Incidentally, the proposed methods have the same stability polynomial which is given as:

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5 \quad (15)$$

The absolute stability region is expressed in figure 1.

Convergence

We will test for the convergence of the derived schemes using the following definitions and theorems.

Definition3.4.1([7])

A numerical method is said to be convergent if for all initial value problems satisfying the hypothesis of the Lipschitz condition given by

$$|f(x,y) - f(x,y^*)| \leq L|y - y^*|$$

where the Lipschitz constant L is denoted by

$$L = \max|f_y(x,y)|.$$

Theorem 3.4.2 ([7])

The necessary and sufficient conditions for a numerical method to be convergent is for it to be consistent and stable.

Definition 3.4.3 ([13])

A numerical method is said to be convergent if it is consistent and has an order greater than one. From the theorem and definitions above, we can conclude that the derived methods are convergent.

Numerical Experiment

In this section, we will compare the numerical performances of the derived methods with some existing numerical methods in the literature. Some linear and nonlinear initial value problems were used to perform this numerical experiment and the methods for comparison are denoted as follows.

- **CM**: The classical 4-stage explicit Runge-Kutta method ,
- **4HERK**: The 4-stage harmonic explicit Runge-Kutta method derived in [19],
- **4MERK**: The 4-stage multiderivative explicit Runge-Kutta method derived in [18],
- **4HRK**: The 4-stage harmonic Runge-Kutta scheme derived in [8] ,
- **4GM**: The Runge-Kutta method with higher order derivative derived in [12].

The basis used in the numerical comparisons is the usual test based on determining the maximum global error in the solutions over the whole integration interval.

Each problem will be integrated with different step sizes and the comparisons is base on the maximum global error versus the step size. These methods were implemented in Wolfram Mathematica.

Problem 1 ([19]). Consider the initial value problem IVP :

$$y' = -y(x), \quad y(0) = 1$$

whose analytic solution is $y(x) = e^{-x}$. The numerical results are presented in Table 1.

TABLE 1: Maximum Global Error Obtained for Problem 1 with *h = 0.01.*

Methods	Maximum Global Error
CM	$3.19642537301E-7$
4GM	$5.99620342820E-6$
4MERK	$3.75025500425E-7$
4GMERK	$1.90708608381E-6$
4HaMERK	$2.00733053111E-6$
4HeMERK	$5.00389914390E-6$
4HERK	$5.00389914390E-6$
4HRK	$2.06380119410E-6$

Problem 2 ([15]). Consider the initial value problem IVP :

$$y' = y(x) - x^2 + 1, \quad y(0) = 0.5$$

whose analytic solution is $y(x) = (x+1)^2 - 0.5e^x$. The numerical results are presented in table 2.

TABLE 2 : Maximum Global Error Obtained for Problem 2 with h = 0.1

Methods	Maximum Global Error
CM	$6.71650908315E-6$
4GM	$6.75559001133E-6$
4MERK	$1.38201581402E-7$
4GMERK	$7.67391103483E-7$
4HaMERK	$9.44100914037E-6$
4HeMERK	$5.33025189037E-7$
4HERK	$6.32919914283E-7$
4HRK	$1.19410392086E-6$

Problem 3 ([4]). Consider the initial value problem IVP :

$$y' = 1 + (y(x))^2, \quad y(0) = 2$$

whose analytic solution is $y(x) = \tan\left(x + \frac{\pi}{4}\right)$. The numerical results are presented in Table 3.

TABLE 3 : Maximum Global Error Obtained for Problem 3 with h = 0.01

Methods	Maximum Global Error
CM	$6.428920542763E-9$
4GM	$8.907743056219E-8$
4MERK	$9.96310643087E-10$
4GMERK	$8.765006321249E-9$
4HaMERK	$9.39640321807E-10$
4HeMERK	$6.096147885433E-9$
4HERK	$9.91477398323E-10$
4HRK	$6.380160194166E-9$

Problem 4 ([13]). Consider the initial value problem IVP :

$$y' = -10(y(x)-1)^2, \quad y(0) = 2$$

whose analytic solution is $y(x) = 1 + \frac{1}{1+10x}$. The numerical results are presented in Table 4.

TABLE 4: Maximum Global Error Obtained for Problem 4 with h = 0.01

Methods	Maximum Global Error
CM	$9.92697302188E-8$
4GM	$9.75420704855E-7$
4MERK	$9.29075373158E-8$
4HaMERK	$7.92629502144E-8$
4GMERK	$5.93873025341E-7$
4HeMERK	$9.19028776297E-8$
4HERK	$9.38900914395E-8$
4HRK	$5.16380849462E-7$

Problem 5 ([21]). Consider the initial value problem IVP :

$$y' = \frac{1}{y}, \quad y(0) = 1$$

whose analytic solution is $y(x) = \sqrt{2x+1}$. The numerical results are presented in Table 5.

TABLE 5 : Maximum Global Error Obtained for Problem 5 with h = 0.1

Methods	Maximum Global Error
CM	$3.67392002725E-9$
4GM	$8.25303293651E-8$
4MERK	$3.35877302755E-9$
4GMERK	$1.19172617002E-8$
4HaMERK	$8.00372912677E-9$
4HeMERK	$3.36383923299E-9$
4HERK	$2.55389914398E-9$
4HRK	$2.06380119410E-8$

Problem 6 ([1]). Consider the initial value problem IVP :

$$y' = xy(x)^2 - y(x), \quad y(0) = 1$$

whose analytic solution is $y(x) = \frac{1}{1+x}$. The numerical results are presented in Table 6.

TABLE 6: Maximum Global Error Obtained for Problem 6 with h = 0.001

Methods	Maximum Global Error
CM	$1.37503472297E-10$
4GM	$1.00048595722E-9$
4MERK	$1.65936104575E-10$
4GMERK	$3.47303720255E-11$
4HaMERK	$4.17595036527E-11$
4HeMERK	$2.75495037334E-10$
4HERK	$5.00389914390E-11$
4HRK	$2.06380119410E-10$

CONCLUSION

In this paper, six (6) problems of different nature particularly linear and nonlinear problems have been numerically solved by the derived methods and compared with some standard numerical methods mentioned from the relevant literature. It may be observed from the tables (1-6) , that the maximum global error produced by some of the derived methods specifically the arithmetic mean viewed higher derivative methods performed better on linear problems while the other types of mean viewed higher derivative methods that is the geometric mean, heronian mean or harmonic mean; performed better on the nonlinear problems. Hence, it would not be out of point to say that the arithmetic mean viewed higher derivative methods performed favorably than the other types of mean viewed higher derivative methods when used to solve some linear initial value problems while other types of mean viewed higher derivative methods performed better than the arithmetic mean viewed higher derivative methods when used to solve non-linear initial value problems. The error analysis: local truncation error, consistency, convergence and stability where investigated wherein they exhibited satisfactory performance. The stability region of the derived methods displayed on Figure 1 revealed that the new methods are stable like every other existing numerical methods in the relevant literature.

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